A two-level Schwarz preconditioner for heterogeneous problems

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1 Introduction

Coarse space correction is essential to achieve algorithmic scalability in domain decomposition methods. Our goal here is to build a robust coarse space for Schwarz–type preconditioners for elliptic problems with highly heterogeneous coefficients when the discontinuities are not just across but also along subdomain interfaces, where classical results break down [3, 6, 15, 9].

In previous work, [7], we proposed the construction of a coarse subspace based on the low-frequency modes associated with the Dirichlet-to-Neumann (DtN) map on each subdomain. A rigorous analysis was recently provided in [2]. Similar ideas to build stable coarse spaces, based on the solution of local eigenvalue problems on entire subdomains, can be found in [4], and even traced back to similar ideas for algebraic multigrid methods in [1]. However, we will argue below that the DtN coarse space presented here is better designed to deal with coefficient variations that are strictly interior to the subdomain, being as robust as, but leading to a smaller dimension than the coarse space analysed in [4].

The robustness result that we obtain, generalizes the classical estimates for overlapping Schwarz methods to the case where the coarse space is richer than just the constant mode per domain [8], or other classical coarse spaces (cf. [15]). The analysis is inspired by that in [4, 13] and crucially uses the framework of weighted Poincaré inequalities, introduced in [12, 10] and successfully applied also to other methods in [11, 14].

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2 Two-level Schwarz method with DtN coarse space

We consider the variational formulation of a second order, elliptic boundary value problem with Dirichlet boundary conditions: Find $u^* \in H_0^1(\Omega)$, for a given domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) and a source term $f \in L_2(\Omega)$, such that

$$a(u^*,\ v) \equiv \int_\Omega \alpha(x)\ \nabla u^* \cdot \nabla v \ = \ \int_\Omega f v \equiv (f,\ v)\ , \quad \forall v \in H_0^1(\Omega), \qquad (1)$$

and the diffusion coefficient $\alpha = \alpha(x)$ is a positive piecewise constant function that may have large variations within Ω.

We consider a discretization of the variational problem (1) with continuous, piecewise linear finite elements (FE). For a shape regular, simplicial triangulation $\mathcal{T}_h$ of Ω, the standard space of continuous and piecewise linear functions (w.r.t $\mathcal{T}_h$) is then denoted by V_h. The subspace of functions from V_h that vanish on the boundary of Ω is denoted by $V_{h,0}$. The discrete FE problem that we want to solve is: Find $u_h \in V_{h,0}$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h,0}. \qquad (2)$$

Given the usual nodal basis $\{\phi_i\}_{i=1}^n$ for $V_{h,0}$ consisting of "hat" functions with $n := \dim(V_{h,0})$, (2) can be compactly written as

$$A\mathbf{u} \ = \ \mathbf{f}, \quad \text{with} \quad A_{ij} := a(\phi_j, \phi_i) \ \text{ and } \ f_i = (f, \phi_i),\ i, j = 1, \ldots, n, \qquad (3)$$

where $\mathbf{u}$ and $\mathbf{f}$ are respectively the vector of coefficients corresponding to the unknown FE function u_h in (2) and to the r.h.s function f.

Two-level Schwarz type methods for (2) are now constructed by choosing an overlapping decomposition $\{\Omega_j\}_{j=1}^J$ of Ω with a subordinate partition of unity $\{\chi_j\}_{j=1}^J$, as well as a suitable coarse subspace $V_H \subset V_{h,0}$. In practice the overlapping subdomains Ω_j can be constructed automatically given the system matrix A by using a graph partitioner, such as METIS, and adding on a number of layers of fine grid elements to the resulting nonoverlapping subdomains. A suitable partition of unity can be constructed from the geometric information of the fine grid. For more details see e.g. [15] or [2]. We assume that each point $x \in \Omega$ is contained in at most N_0 subdomains Ω_j.

The crucial ingredient to obtain robust two-level methods for problems with heterogeneous coefficients is the choice of coarse space $V_H \subset V_{h,0}$. Let us assume for the moment that we have such a space V_H and a restriction operator R_0 from $V_{h,0}$ to V_H and define restriction operators R_j from functions in $V_{h,0}$ to functions in $V_{h,0}(\Omega_j)$, or from vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^{\dim V_{h,0}(\Omega_j)}$, by setting $(R_j u)(x_i) = u(x_i)$ for every grid point $x_i \in \Omega_j$. The two-level overlapping additive Schwarz preconditioner for (3) is then simply

$$M_{AS,2}^{-1} = \sum_{j=0}^{J} R_j^T A_j^{-1} R_j \quad \text{where} \quad A_j := R_j A R_j^T,\ j = 0, \ldots, J. \qquad (4)$$

In the classical algorithm V_H consists simply of FEs on a coarser triangulation $\mathcal{T}_H$ of Ω and R_H is the canonical restriction from $V_{h,0}$ to V_H, leading to a fully scalable iterative method with respect to mesh/problem size (provided the overlap size is proportional to the coarse mesh size H). However, unfortunately this preconditioner is not robust to strong variations in the coefficient α. We will now present a new, completely local approach to construct a robust coarse space, as well as an associated restriction operator using eigenvectors of local Dirichlet-to-Neumann maps, proposed in [7].

We start by constructing suitable local functions on each subdomain Ω_j that will then be used to construct a basis for V_H. To this end, let us fix $j \in \{1, \ldots, J\}$ and first consider at the continuous level the Dirichlet-to-Neumann map DtN_j on the boundary of Ω_j. Let $\Gamma_j := \partial\Omega_j$ and let $v_\Gamma : \Gamma_j \to \mathbb{R}$ be a given function, such that $v_\Gamma|_{\partial\Omega} = 0$ if $\Gamma_j \cap \partial\Omega \neq \emptyset$. We define

$$\mathrm{DtN}_j(v_\Gamma) \;:=\; \alpha \frac{\partial v}{\partial \nu_j}\bigg|_{\Gamma_j} \,,$$

where ν_j is the unit outward normal to Ω_j on Γ_j, and v satisfies

$$-\mathrm{div}(\alpha \nabla v) = \; 0 \text{ in } \Omega_j, \quad v = \; v_\Gamma \text{ on } \Gamma \,. \tag{5}$$

The function v is the α–harmonic extension of the boundary data v_Γ to the interior of Ω_j.

To construct the (local) coarse basis functions, we now find the low frequency modes of the Dirichlet-to-Neumann operator DtN_j with respect to the weighted L_2–norm on Γ_j, i.e. the smallest eigenvalues of

$$\mathrm{DtN}_j(v_\Gamma^{(j)}) \;=\; \lambda^{(j)}\, \alpha v_\Gamma^{(j)} \,. \tag{6}$$

Then we extend each of these modes $v_\Gamma^{(j)}$ α–harmonically to the whole domain and let $v^{(j)}$ be its extension. This is equivalent to the Steklov eigenvalue problem of looking for the pair $(v^{(j)}, \lambda^{(j)})$ which satisfies:

$$-\mathrm{div}(\alpha \nabla v^{(j)}) = \; 0 \text{ in } \Omega_j \quad \text{and} \quad \alpha \frac{\partial v^{(j)}}{\partial \nu_j} = \lambda\, \alpha v^{(j)} \text{ on } \Gamma_j. \tag{7}$$

The variational formulation of (7) is to find $(v^{(j)}, \lambda^{(j)}) \in H^1(\Omega_j) \times \mathbb{R}$ such that

$$\int_{\Omega_j} \alpha \nabla v^{(j)} \cdot \nabla w = \lambda^{(j)} \int_{\Gamma_j} \mathrm{tr}_j \alpha\, v^{(j)}\, w\,, \quad \forall w \in H^1(\Omega_j), \tag{8}$$

where $\mathrm{tr}_j \alpha(x) := \lim_{y \in \Omega_j \to x} \alpha(y)$. To discretize this generalized eigenvalue problem, we consider for all $v, w \in H^1(\Omega_j)$ the bilinear forms

$$a_j(v,w) := \int_{\Omega_j} \alpha \nabla v \cdot \nabla w \quad \text{and} \quad m_j(v,w) \;:=\; \int_{\Gamma_j} \mathrm{tr}_j \alpha v w$$

and restrict (8) to the FE space $V_h(\Omega_j)$. The coefficient matrices associated with the variational forms a_j and m_j are

$$A_{kl}^{(j)} := \int_{\Omega_j} \alpha \nabla\phi_k \cdot \nabla\phi_l \quad \text{and} \quad M_{kl}^{(j)} := \int_{\Gamma_j} \mathrm{tr}_j \alpha \phi_k \, \phi_l,$$

where ϕ_k and ϕ_l are any two nodal basis functions for $V_h(\Omega_j)$ associated with vertices of $\mathcal{T}_h$ contained in $\overline{\Omega}_j$. Then the FE approximation to (8) in matrix notation is

$$A^{(j)}\mathbf{v}^{(j)} = \lambda^{(j)} M^{(j)} \mathbf{v}^{(j)} \tag{9}$$

where $\mathbf{v}^{(j)} \in \mathbb{R}^{n_j}$, $n_j := \dim V_h(\Omega_j)$, denotes the degrees of freedom of the FE approximation to $v^{(j)}$ in $V_h(\Omega_j)$.

Let the n_j eigenpairs $(\lambda_\ell^{(j)}, \mathbf{v}_\ell)_{\ell=1}^{n_j}$ corresponding to (9) be numbered in increasing order of $\lambda_\ell^{(j)}$. Since $M_{kl}^{(j)} \neq 0$ only if ϕ_k and ϕ_l are associated with the n_Γ vertices of $\mathcal{T}_h$ that lie on Γ_j, it is easy to see that at most n_Γ of the eigenvalues $\lambda_\ell^{(j)}$ are finite. Moreover, the smallest eigenvalue $\lambda_1^{(j)} = 0$ with constant eigenvector and the set of eigenvectors $\{\mathbf{v}_\ell\}_{\ell=1}^{n_j}$ can be chosen so that they are $A^{(j)}$–orthonormal. The local coarse space is now defined as the span of the FE functions $v_\ell^{(j)} \in V_h(\Omega_j)$, $\ell \leq m_j \leq n_\Gamma$, corresponding to the first m_j eigenpairs of (9). For each subdomain Ω_j, we choose the value of m_j such that $\lambda_\ell^{(j)} < \mathrm{diam}(\Omega_j)^{-1}$, for all $\ell \leq m_j$, and $\lambda_{m_j+1}^{(j)} \geq \mathrm{diam}(\Omega_j)^{-1}$. We will see in the analysis in the next section why this is a sensible choice.

Using the partition of unity $\{\chi_j\}_{j=1}^J$, we now combine the local basis functions constructed in the previous section to obtain a conforming coarse space $V_H \subset V_{h,0}$ on all of Ω. The new coarse space is defined as

$$V_H \;:=\; \mathrm{span}\left\{ I_h\Big(\chi_j v_\ell^{(j)}\Big) : 1 \leq j \leq J \text{ and } 1 \leq \ell \leq m_j \right\}, \tag{10}$$

where I_h is the standard nodal interpolant onto $V_{h,0}(\Omega)$. The dimension of V_H is $\sum_{j=1}^J m_j$. By construction each of the functions $I_h\big(\chi_j v_\ell^{(j)}\big) \in V_{h_0}$, so that as required $V_H \subset V_{h,0}$. The transfer operator R_0 from V_{h_0} to V_H is defined in a canonical way by setting $R_0^T u_H(x_i) = u_H(x_i)$, for all $u_H \in V_H$ and for all vertices x_i of $\mathcal{T}_h$.

We will see in the next section that under some mild assumptions on the variability of α this choice of coarse space leads to a scalable and coefficient-robust domain decomposition method with supporting theory.

3 Conditioning analysis

To analyse this method let us first define the boundary layer Ω_j° for each Ω_j that is overlapped by neighbouring domains, i.e.

$$\Omega_j^\circ := \{x \in \Omega_j : \chi_j(x) < 1\}.$$

We assume that this layer is uniformly of width $\geq \delta_j$, in the sense that it can be subdivided into shape regular regions of diameter δ_j, and that the triangulation $\mathcal{T}_h$ resolves it. This also guarantees that it is possible to find a partition of unity such that $|\chi_j| = \mathcal{O}(1)$ and $|\nabla \chi_j| = \mathcal{O}(\delta_j^{-1})$.

We now state the key assumption on the coefficient distribution $\alpha(x)$.

Assumption 1 We assume that, for each $j = 1, \ldots, J$, there exists a set $X_j \subset \Gamma_j$ (not necessarily connected) such that (i) $\max_{x,y \in X_k} \frac{\alpha(x)}{\alpha(y)} = \mathcal{O}(1)$ and (ii) there exists a path P_y from each $y \in \Omega_j$ to X_j, such that $\alpha(x)$ is an increasing function along P_y (from y to X_j).

Lemma 1 (weighted Poincaré inequality [10]). *Let Assumption 1 hold.*

$$\int_{\Omega_j^\circ} \alpha |v - \overline{v}^{X_j}|^2 \ \leq \ C_P\, \delta_j \int_{\Omega_j^\circ} \alpha |\nabla v|^2, \qquad \textit{for all} \ \ v \in V_h(\Omega_j),$$

where $\overline{v}^{X_j} := \frac{1}{|X_j|} \int_{X_j} v$.

Remark 1. Note that Assumption 1 is related to the classical notion of quasi-monotonicity coined in [3]. It ensures that the constant C_P in the Poincaré-type inequality in Lemma 1, as well as all the other (hidden) constants below are independent of the values of the coefficient function $\alpha(x)$. The constants may however depend logarithmically or linearly on δ_j/h. This depends on the geometry and shape of the paths P_y and on the size and shape of the set X_j. For more details see [2] and [12, 10].

The following proposition is the central result in our analysis. It proves the stability and a weak approximation property for a local projection onto the span of the first m_j eigenvectors.

Proposition 1. *Let Assumption 1 hold, and for any $u \in V_h(\Omega_j)$, define the projection $\Pi_j u := \sum_{\ell=1}^{m_j} a_j(v_\ell^{(j)}, u)\, v_\ell^{(j)}$. Then*

$$|\Pi_j u|_{a,\Omega_j} \leq |u|_{a,\Omega_j} \quad \textit{and} \tag{11}$$

$$\|u - \Pi_j u\|_{0,\alpha,\Omega_j^\circ} \lesssim \sqrt{c_j(m_j)}\, \delta_j\, |u|_{a,\Omega_j}\, . \tag{12}$$

where $c_j(m_j) := C_P^2 + \left(\delta_j \lambda_{m_j+1}^{(j)}\right)^{-1}$.

Proof. Theorem 3.2 in [2].

As usual (cf. [15]), the following condition number bound can then be obtained via abstract Schwarz theory by constructing a stable splitting.

Theorem 2. *Let Assumption 1 be satisfied. Then the condition number of the two-level Schwarz algorithm with the coarse space V_H based on local DtN maps and defined in* (10) *can be bounded by*

$$\kappa(M_{AS,2}^{-1}A) \;\lesssim\; \max_{j=1}^{J}\{c_j(m_j)\} \;\lesssim\; C_P^2 + \max_{j=1}^{J}\left(\delta_j \lambda_{m_j+1}^{(j)}\right)^{-1}.$$

The hidden constant is independent of h, δ_j, $\operatorname{diam}(\Omega_j)$*, and α.*

Proof. This is Theorem 3.5 in [2]. The stable splitting for a function $u \in V_{h,0}$ is constructed using the projections Π_j, $j = 1, \ldots, J$, in Proposition 1 to define the coarse quasi-interpolant

$$u_0 := I_h\left(\sum_{j=1}^{J} \chi_j \Pi_j u|_{\Omega_j}\right) \in V_H\,. \tag{13}$$

If we now choose $u_j := I_h(\chi_j(u - \Pi_j u)) \in V_{h,0}(\Omega_j)$, then

$$u = \sum_{j=0}^{J} u_j \quad \text{and} \quad \sum_{j=0}^{J} \int_\Omega \alpha|\nabla u_j|^2 \;\lesssim\; \max_{j=1}^{J}\{c_j(m_j)\} \int_\Omega \alpha|\nabla u|^2$$

For details see [2].

Remark 2. Note that by choosing the number m_j of modes per subdomain such that $\lambda_{m_j+1}^{(j)} \geq \operatorname{diam}(\Omega_j)^{-1}$, as stated in Section 2, we have

$$\kappa(M_{AS,1}^{-1}A) \lesssim \left(C_P^2 + \max_{j=1}^{J} \frac{\operatorname{diam}(\Omega_j)}{\delta_j}\right).$$

Hence, provided the constant C_P is uniformly bounded, independently of any jumps in the coefficients, we retrieve the classical estimate for the two-level additive Schwarz method independently of any variations of coefficients across or along subdomain boundaries.

4 Numerical results

We choose $\Omega = (0,1)^2$ and discretize (1) on regular grid with $m \times m$ elements. We apply a homogeneous Dirichlet boundary condition $u = 0$ on the left hand side boundary and homoegenous Neumann boundary conditions $\frac{\partial u}{\partial \nu} = 0$ on the remainder. We use the METIS partitioner to split the domain into 16 irregular subdomains as shown in Figure 1. Then we construct the overlapping partition using Freefem++ [5] by extending each subdomain by one layer of elements on all or the boundary.

As the coarse space we use the DtN coarse space described in Section 2 with m_j chosen such that $\lambda_{m_j}^{(j)} < \operatorname{diam}(\Omega_j)^{-1} \leq \lambda_{m_j+1}^{(j)}$, for all $j = 1, \ldots, 16$ (labelled `D2N` below). We compare this preconditioner with the one-level additive Schwarz method (labelled `NONE` below) and the two-level method with partition of unity coarse space, i.e. choosing $m_j = 1$, for all $j = 1, \dots, 16$ (labelled `POU` below). To confirm in some sense the optimality of our choice for m_j, we also include results with the DtN coarse space choosing $m_j + 1$ and $\max\{1, m_j - 1\}$ basis functions per subdomain (labelled `D2N+` and `D2N-`, respectively). We use the preconditioners within a conjugate gradient iteration and terminate when the residual has been reduced by a factor 10^{-6}.

In the first test case, we choose $m = 80$ and α to be a realization of a log-normal distribution with exponential covariance function (variance $\sigma^2 = 4$ and correlation length $\lambda = 4/m$) and mean$(\log \alpha) = 3$ (cf. Figure **??**).

In Figure 3 we plot $\|u - \bar{u}\|_\infty$ (where $\bar{u}$ is the solution of (3) obtained via a direct solver) against the iteration count. We compare three methods:

AS : the one level preconditioner $M_{AS,1}^{-1} = \sum_{j=1}^{J} R_j^T A_j^{-1} R_j$.
AS + Z_{NICO} : the two level preconditioner defined by (4) with a coarse grid which consists simply of constant functions on each subdomain weighted by a partition of unity.
AS + Z_{D2N} : the two level preconditioner defined by (4) with the new coarse grid we have introduced.

The AS and AS + Z_{NICO} methods require roughly the same number of iterations (89 versus 92 iterations) whereas the new AS + Z_{D2N} stands out reducing the number of iterations to 38. Finally, in Figure 4 we show that the criterion for the number m_j of eigenmodes that we select in each subdomain is somewhat optimal since adding one has hardly any impact on performance while removing one has strong negative impact.

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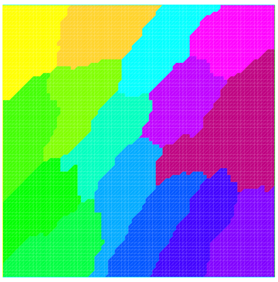

Fig. 1 16 subdomains

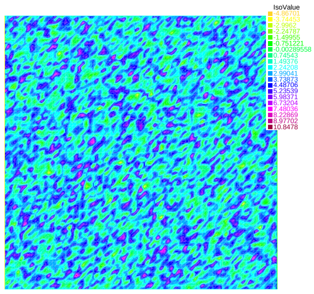

Fig. 2 $\log(\alpha)$ $(0 < \alpha < 5 \cdot 10^7)$

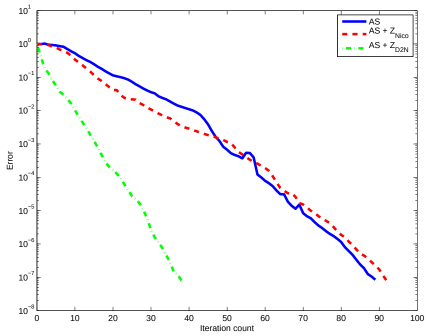

Fig. 3 Convergence

	AS	AS + Z_{NICO}	AS + Z_{D2N}
$\max(m_j - 1, 1)$			50
m_j	89	92	38
$m_j + 1$			36

Fig. 4 Optimality of the criterion – Number of iterations needed when we add or remove one mode per subdomain as compared to the number m_j given by the automatic method

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